FretFitter
Temperament and Fretted Clavichords

Alan Shepherd and Gregor Bergmann

Change History (including changes to the Excel file)

Version	Author	Change
V01.00	Shepherd	First Release.

Table of Contents

1 Introduction

1.1 Acknowledgements

We would like to thank Hans Eugen Frischknecht and Jakob Schmidt for their kind permission to use the temperament data, hosted by the Bund Deutscher Orgelbaumeister (Federation of German Master Organ Builders) [BDO].

1.2 Overview

This paper is a coming together of one author's (Shepherd) attempts to understand musical temperament and the other's (Bergmann) need to calculate the distances between tangents on fretted clavichords and to compare the fretting of existing clavichords with known temperaments. The original idea to produce a calculator and data base of clavichords is Bergmann's.

The main product of this work is a Microsoft® Excel® file , FretFitter_V01.00.xlsx, with various tables:

- To define temperaments and see their characteristics
- To define the dimensions of frets, strings or pipes
- To calculate fret distances for a given temperament
- To compare the fret intervals of an instrument with temperaments
- To find the temperaments that fit best to an instrument or self-defined temperament.

The worksheets fall into the following groups:

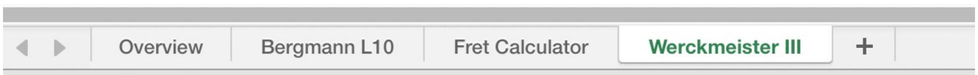
- Explanatory Worksheets (chapter 2) – the navigation overview and some data on commas and cents.
- Calculation Worksheets (chapter 3) – for calculating the fretting and finding the temperaments that best fit a given instrument or temperament.
- Fretting Worksheets (chapter 3.3) – the measured frets of existing instruments and comparison to up to four temperaments.
- Temperament Worksheets (chapter 5) – the data for analysing temperaments or instruments.
- Template Worksheets (chapter 6) – templates for copying to create new fretting and temperament worksheets.

The sources used are given in chapter 7.

There are further appendices with more detailed information about the formulae and spreadsheet interna.

1.3 Note Naming Conventions

The worksheets in the file are referred to here by their names in single quotes, e.g. 'Werckmeister III' shown in Excel below the sheets:



(The lengths of the names is restricted to 31 characters so some are abbreviated.)

Note names

There are various conventions for naming the notes of the scale in different octaves, e.g.

Scientific	Helmholtz	English
C0	C,,	CCC
C1	C,	CC
C2	C	C
C3	c	c
C4	c’	cc
C5	c’’	ccc
C6	c’’’	cccc
C7	c’’’’	ccccc
C8		
C9		

We use the Helmholtz column but denote the octave of C as “c oct.” because Excel is not case sensitive in some functions and so would not differentiate “C” from “c”.

For the chromatic notes we always give the sharp rather than the flat, so A# and not Bb.

All notes are reduced to the middle octave C – c as we are mainly interested in relative frequencies or lengths, usually in cents which are independent of the octave.

1.4 Tuning and Temperament

It is not the intention of this paper to explain the theory of tuning and temperament but we do try to give enough information for users to verify the calculations. It is assumed that the reader is familiar with the concepts and terms such as comma and cent. The Bibliography gives some sources for this knowledge,.e.g. [Benson, 2007], [Duffin, 2007]. The most important equations are given in Appendix A for reference.

Since most sources show the tuning starting at C, we do the same. Details of calculating the starting note C from the frequency of A are given in appendix B.4.4. Some instructions for modifying the scales to start at A are given in section 6.3.

An important historical work is [Werckmeister, 1691] which describes how to obtain tunings and temperaments with a monochord. The main diagram was given in a separate engraving known as the “Kupferstich”. This is shown in the sheet ‘Monochord’.

Figure 1-1 shows some of the main originators of temperaments and their works. Each person is represented as a timeline between their birth and death. Their books are represented below this, with their age at the time of publication shown above the timeline. The dates are shown on the horizontal axis at the bottom. This spans the overall timeframe, with additional vertical lines to show the starts of years for orientation.

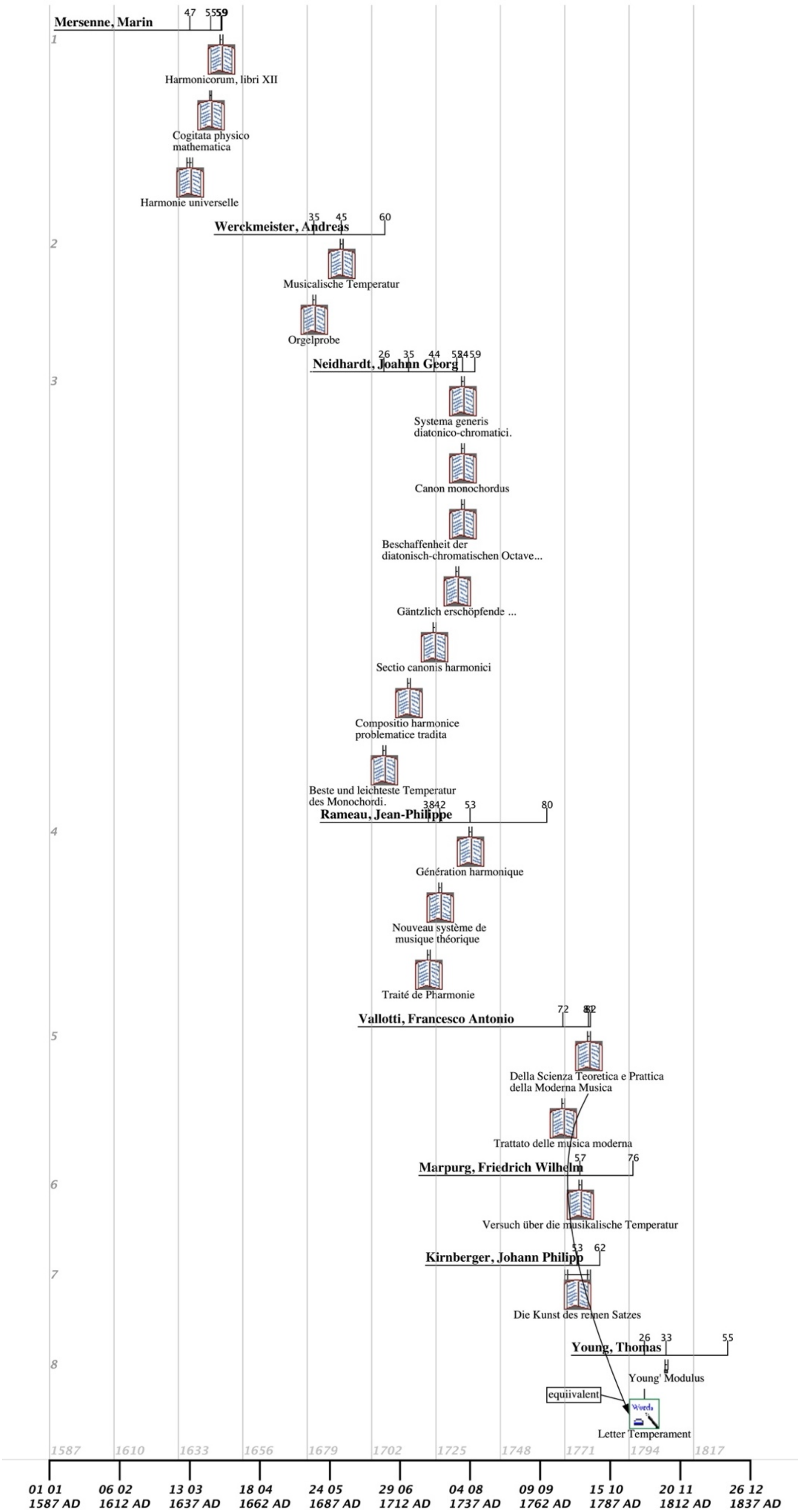


Figure 1-1 Timeline of Historical Figures

1.5 Temperament Sources

There is a multitude of sources for temperaments, all using different conventions. Where possible we prefer to work directly from the original historical source. Some examples of online sources are compared here. Note that many are under copyright, so the conditions must be adhered to for publication or commercial use.

The basic differences between sources are, for example:

- The use of Pythagorean comma, Syntonic comma, fractions thereof, or cents.
- Giving differences to equal temperament or just temperament.
- Giving differences to the just interval from the previous note or from the note in the equal or just scale.
- Based on starting note C or A.
- Ordered by fifths (C-G-D-… or by scale C-C#-D-…).
- The precision given, i.e. the number of decimal places.

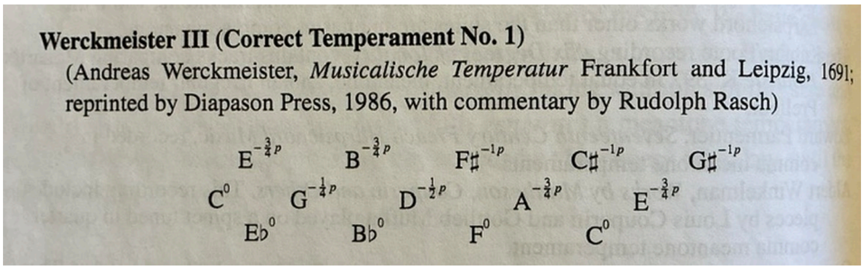
For the original [Werckmeister, 1691] see section 5.17 and the Eitz representaion from [Benson, 2007] (Figure 1-2).

The German society of Master Organ Builders [BDO] gives a list of temperaments and a graphical representation of each. The links within the list contain the differences to Equal Temperament in cents ordered in fifths to three decimal places. The graphics give the differences in whole cents and in some cases to two decimal places. There are lists of cents from C, cents between fifths, cents between thirds. This is the source we use for reference.

[Huygens-Fokker] provides a collection of files for over 4000 scales. These give the notes in scale order in a mixture of fractional and decimal values. Fractions, e.g. 4/3 are the ratio of the note from C, so a fraction a/b would give the $note = C * \frac{a}{b}$. Decimals give the cents from C, so a decimal d would give the $note = C * 2^{d/1200}$. The values from the file cannot be directly used in Excel.

The makers of tuning apps also provide temperaments, usually within the app.[NetCat] is an example of one that provides some data online in cents from ET in scale order. However, their data does not match the other sources.

Figure 1-2 shows an example. They are all in agreement except for [NetCat]. See Appendix C.3 for an Excel sheet with the comparison.


Werckmeister III (Correct Temperament No. 1)

(Andreas Werckmeister, *Musicalische Temperatur* Frankfort and Leipzig, 1691; reprinted by Diapason Press, 1986, with commentary by Rudolph Rasch)

A Werckmeister III from Benson

Tuning Defined in Fifths						
Note	Relative	Cumulative	F Hz	Cents btw. Fifths	Octaves down	F within octave
C	0	0	263,4042326		0	263,40423
G	-1	-1	393,77009	696,09000	0	393,77009
D	-1	-2	588,65752	696,09000	1	294,32876
A	-1	-3	880,00000	696,09000	1	440,00000
E	0	-3	1320,00000	701,95500	2	330,00000
B	0	-3	1980,00000	701,95500	2	495,00000
F#	-1	-4	2959,95538	696,09000	3	369,99442
C#	0	-4	4439,93307	701,95500	4	277,49582
G#	0	-4	6659,89961	701,95500	4	416,24373
D#	0	-4	9989,84941	701,95500	5	312,18279
A#	0	-4	14984,77412	701,95500	5	468,27419
F	0	-4	22477,16118	701,95500	6	351,20564
c oct.	0	-4	33715,74177	701,95500	6	526,80847

Sc	
Note Scale Order	F Scale Order
C	263,40423
C#	277,49582
D	294,32876
D#	312,18279
E	330,00000
F	351,20564
F#	369,99442
G	393,77009
G#	416,24373
A	440,00000
A#	468,27419
B	495,00000
c oct.	526,80847

B Werckmeister III from Original

Tuning Defined in Cents Relative to ET				
Note Scale Order	ET Frequency Hz	Difference to ET Scale in Cents	Resulting F	Monochord String Length
C	263,40423	0	263,40423	3600,00
C#	279,06706	-9,775	277,49582	3417,19
D	295,66125	-7,82	294,32876	3221,76
D#	313,24219	-5,865	312,18279	3037,50
E	331,86854	-9,775	330,00000	2873,50
F	351,60247	-1,955	351,20564	2700,00
F#	372,50984	-11,73	369,99442	2562,89
G	394,66042	-3,91	393,77009	2408,14
G#	418,12815	-7,82	416,24373	2278,12
A	442,99135	-11,73	440,00000	2155,13
A#	469,33298	-3,91	468,27419	2025,00
B	497,24098	-7,82	495,00000	1915,67
c oct.	526,80846	0	526,80846	1800,00

C Werckmeister III from BDO Links

Tuning Defined in Cents from C					
Note Scale Order	Cents from C	F Scale Order	Relative Frequency Cents	Monochord String Length	File
C	0,00000	263,40423		3600,00	
C#	1,05350	277,49582	90,225	3417,19	256/243
D	192,18000	294,32876	101,955	3221,76	192,18
D#	1,18519	312,18280	101,955	3037,50	32/27
E	390,22500	330,00000	96,090	2873,50	390,225
F	1,33333	351,20564	107,820	2700,00	4/3
F#	1,40466	369,99442	90,225	2562,89	1024/729
G	696,09000	393,77009	107,820	2408,14	696,09
G#	1,58025	416,24373	96,090	2278,13	128/81
A	888,26999	440,00000	96,090	2155,13	888,26999
A#	1,77778	468,27419	107,820	2025,00	16/9
B	1092,18000	495,00000	96,090	1915,67	1092,18
c oct.	2,00000	526,80847	107,820	1800,00	2/1

D Werckmeister III from Scala File

Tuning Defined in Cents Relative to ET				
Note Scale Order	ET Frequency Hz	Difference to ET Scale in Cents	Resulting F	Monochord String Length
C	261,62557	0,00000	261,62557	3600,00
C#	277,18263	-3,91000	276,55732	3405,63
D	293,66477	3,91000	294,32876	3200,00
D#	311,12698	0,00000	311,12698	3027,23
E	329,62756	-3,91000	328,88393	2863,78
F	349,22823	3,91000	350,01786	2690,87
F#	369,99442	0,00000	369,99442	2545,58
G	391,99544	1,95500	392,43835	2400,00
G#	415,30470	-7,82000	413,43299	2278,12
A	440,00000	0,00000	440,00000	2140,57
A#	466,16376	1,95500	466,69048	2018,15
B	493,88330	-1,95500	493,32590	1909,19
c oct.	523,25113	0,00000	523,25113	1800,00

E Werckmeister III from NetCat

Figure 1-2 Examples of Representations of Werckmeister III Temperament

1.6 Beats

Tuning two strings to a unison is often described by listening for the beats or changes in perceived amplitude. For two pure sine waves of frequencies f_1 and f_2 Hz, the beat frequency will be f_2-f_1 beats per second. When the unison is perfect there will be no beats. Beats between intervals such as a fifth are also perceptible and are often recommended for tuning tempered intervals.

However, since other intervals than the unison can cause beats, and musical instruments do not produce pure sine waves but waves rich in harmonics, there can be beats between various harmonics.

Also, unlike cents, the frequency of the beats depends on the octave in which the notes are played.

The Harmonics and the Octave Transposer sections of the temperament sheets may be useful to see which harmonics might beat with other notes.

This becomes very complex and error-prone, so we do not use beats in this context.

1.7 Hints for Using Excel

For those who are not experts in using Microsoft® Excel® we provide some hints on how to navigate the high number of worksheets involved. The commands are the same on Microsoft® Windows® or Apple® Macintosh®, except that the Control-key (Ctrl) on Windows is the Command-key on Mac. We denote this here as Cmd/Ctrl. (The workbooks should also work with Excel on tablets with iOS® or Android®, although some functions may not be available).

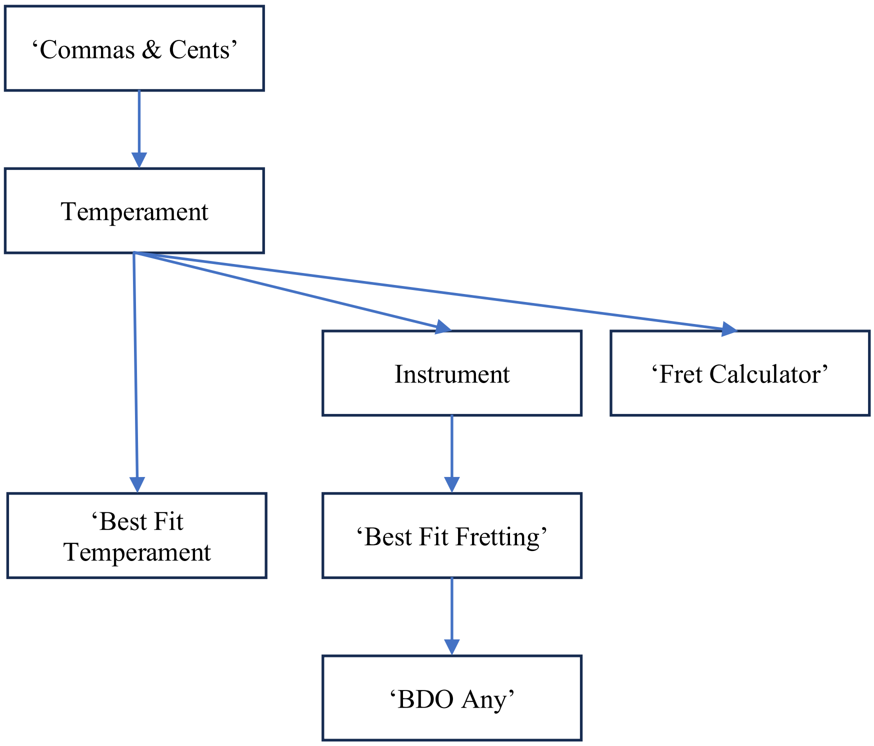
The procedures should be the same in other languages than English with the items in the same places. Further help can be obtained by searching the Internet.

In Excel terminology a “workbook” is a file (of type .xlsx) that contains “worksheets”, which are individual spreadsheet tables.

Note: some functions do not work with other programs such as LibreOffice® (The Document Foundation TDF®) or Numbers® (Apple Inc®), for example, coloured tabs of worksheets, dynamic titles of graphics, protection of cells.

1.7.1 Worksheet Dependencies

The data in some sheets is used in other sheets, so the following worksheets must be within the same workbook:



Temperament worksheets need the 'Commas & Cents' worksheet, as this defines the choice of units.

Instrument worksheets need one or more temperament worksheets for comparison.

The 'Fret Calculator' needs one or more temperament worksheets.

The 'Best Fit Fretting' worksheet needs at least one instrument worksheet to fit to.

The 'Best Fit Temperament' needs at least one temperament worksheet to fit to.

The 'BDO Any' Worksheet needs the 'Best Fit Fretting' worksheet, as this defines the BDO temperaments.

1.7.2 Inserting and Deleting Worksheets

New worksheets should normally be created from the existing template worksheets – see section 1.7.6.

There is a large number of worksheets in the workbook file, so it may be useful to delete those not needed for the current project. Worksheets are deleted by a right-click on the worksheet tab.

Always make a copy of the workbook file before deleting worksheets.

Keep the Overview sheet up to date.

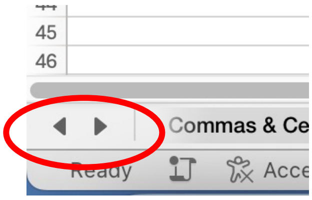
Bear the dependencies given in section 1.7.1 in mind when deleting unneeded worksheets.

1.7.3 Navigation

The tabs of all the worksheets probably do not fit on the bottom of the screen. You can switch between sheets as follows:

With the Overview sheet. This has a table of contents where you can click to go to any sheet. All the sheets have a “Back to Overview” - click this to return to the overview.

With the arrows at the bottom left of the window:

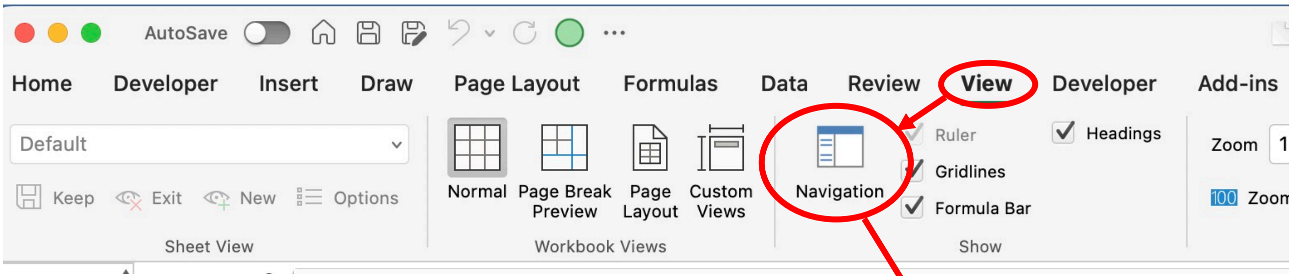


Click on an arrow to move the list left or right

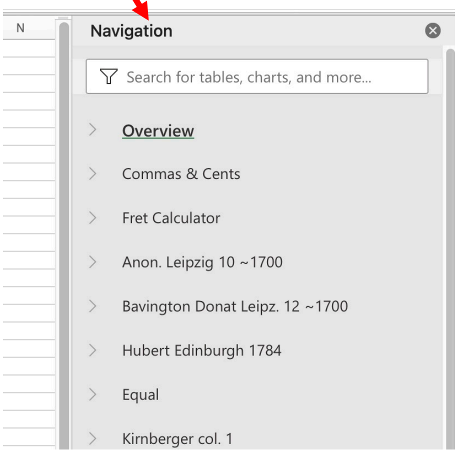
Cmd/Ctrl-click moves to the first or last tab.

A right-click shows a list of tabs – click on the one to open.

A permanent navigation list can be obtained from the View – Navigation menu:

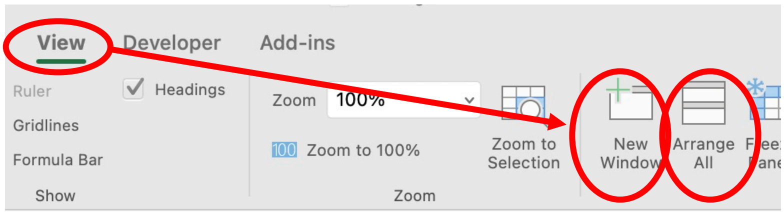


You will probably want to click on any down-facing arrow here to collapse the contents of the current sheet.

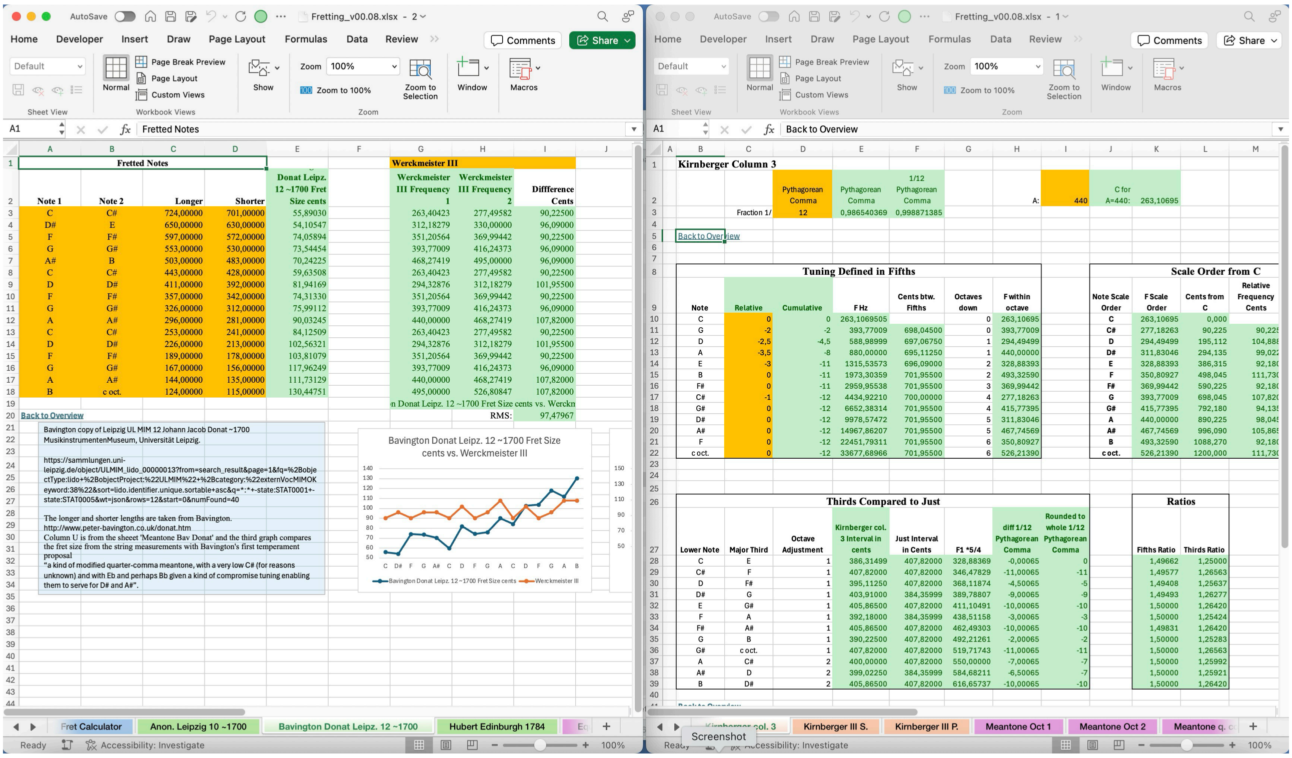


On the keyboard you can also use Ctrl-Page Up or Ctrl-Page Down (on Windows) or Option-Right-Arrow and Option-Left-Arrow (on Mac) to go to the previous or next worksheet.

1.7.4 Viewing two Sheets at Once

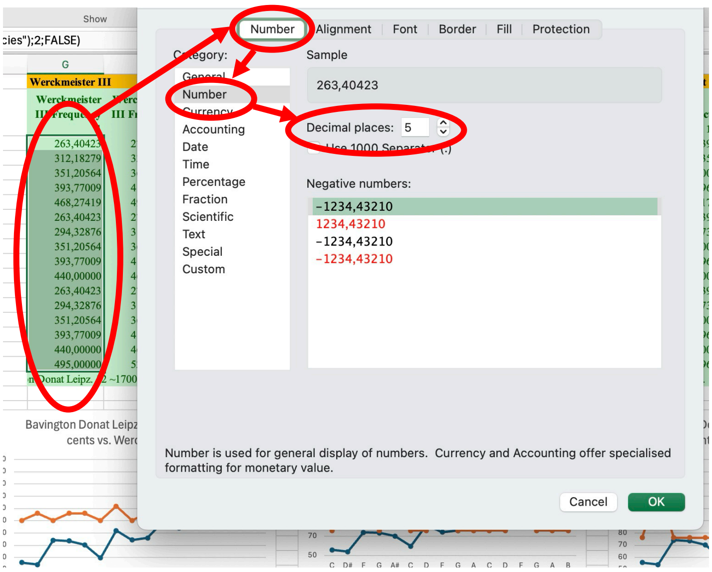


To see two (or more) sheets on the screen, e.g. for comparison, click the menu View – New Window and then Arrange All. Choose vertically or horizontally. To get back to a single window, close the others.



1.7.5 Changing Number Formats

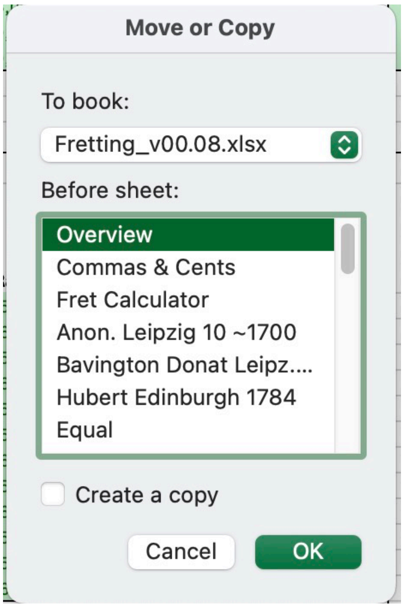
Numbers are mostly shown to two decimal places as some people find 3 or 5 places overwhelming. Excel always maintains the full precision, even when not all places are shown.



To change the format, select one or several numbers, right-click and select "Format Cells …", or use the keyboard shortcut Cmd/Ctrl-1. In the resulting window select the Number tab, the Number category and then set the decimal places.

1.7.6 Copying and Moving Worksheets

Worksheets can by moved or copied, either within the same workbook or between open workbooks. Using multiple workbooks would be one way of reducing the number of worksheets in a single workbook. Copy is important for creating new worksheets from the templates or from other worksheets.



To move or copy, right-click on the tab of the workbook to be moved or copied and select “Move or Copy”. In the dialogue box:

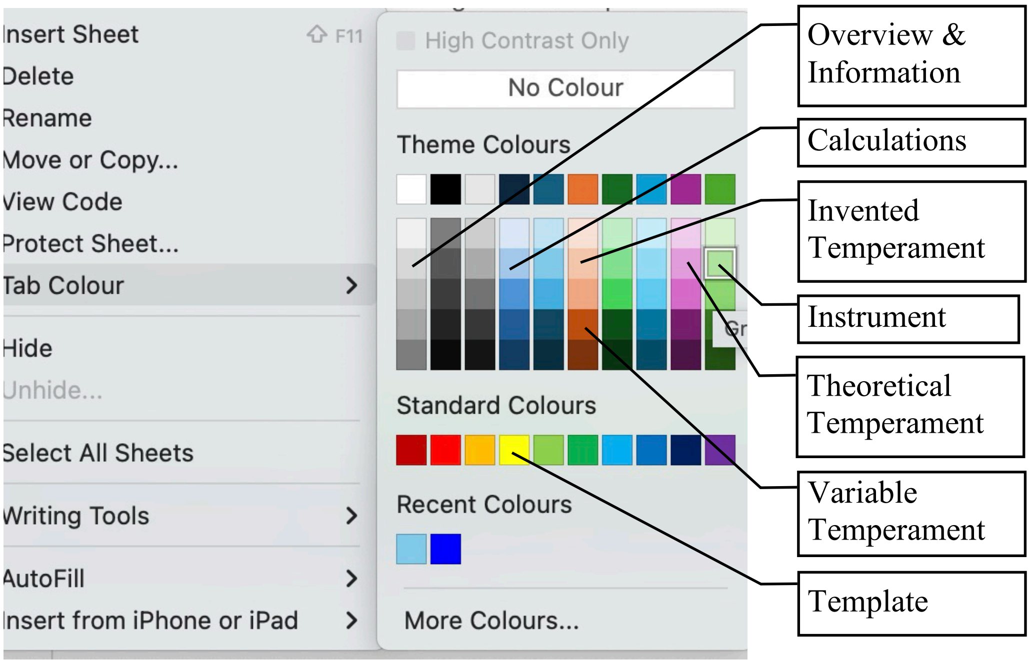
To copy to another workbook, select it in the drop-down box “To book:”. (The workbook must be already open.)

To copy rather than move, check the box “Create a copy”. (If you forget this, the worksheet will be moved, i.e. removed from the original place.)

Be sure to take the dependencies into account when copying or moving worksheets between workbooks – see section 1.7.1.

1.7.7 Changing a Worksheet Colour

Right click on the worksheet’s tab and click “Tab Colour”. We use the following convention:

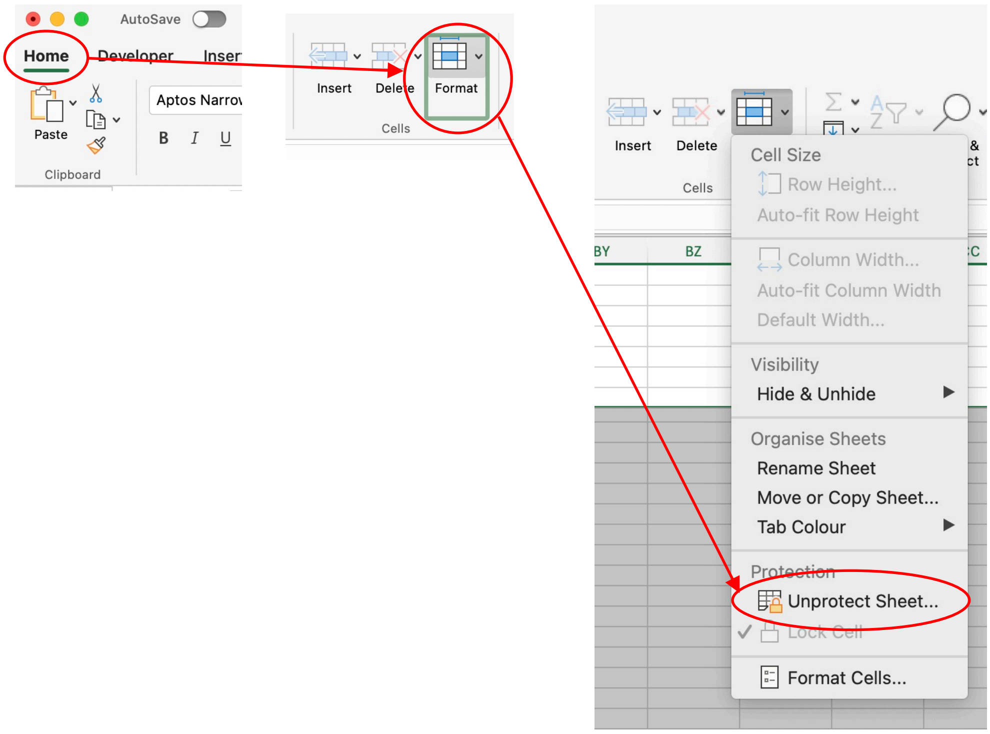


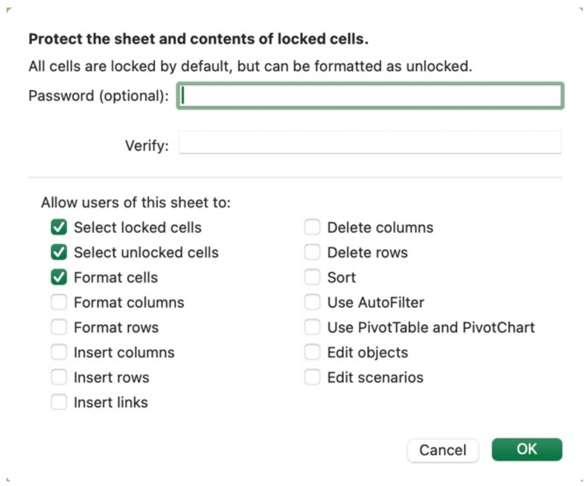
1.7.8 Protection

To prevent accidental changes to the formulae, the worksheets are protected. Only the fields with orange backgrounds can be changed.

Some functions can only be performed after removing the protection, e.g. inserting or deleting rows, sorting the 'Best Fit' worksheets.

To remove the protection, right-click on the worksheet tab and select Unprotect sheet. Alternatively, use the menu Home in the Cells group, click Format and Unprotect Sheet.





To reinstate the protection, click Protect Sheet in the same way and set the options as shown.

The option for "Format cells" is ticked so that the decimal places in the numbers can be changed.

Remember to leave the worksheet protected when finished.

Protected sheets are marked with a padlock symbol in the title tab.

To unlock or lock an individual cell click Lock Cell in the Format menu, its current state is shown by a tick when locked.

2 Explanatory Worksheets

2.1 Overview Sheet

This gives the authors and refers to this document.

The overview sheet indexes the other sheets for easy access. The other sheets contain a link back to the overview for further navigation.

2.2 Commas and Cents

This sheet gives the values of fractions of commas in cents for Pythagorean and Syntonic commas. This is convenient as temperaments are variously given in the literature in cents, commas or fractions of commas.

The schisma is also given in various forms (see Appendix A for the formulae):

- As a real[1] number
- In cents
- In Pythagorean commas
- In Syntonic commas
- As fractions of Pythagorean and Syntonic commas

[1] "Real number" is the mathematical term for numbers represented as decimals.

3 Calculation Worksheets

3.1 Fret Calculator

The first three sections of this sheet are used to perform ad hoc calculations of fret sizes:

- Enter the longer string length and the cents difference to obtain the shorter string length.
- Enter the shorter string length and the cents difference to obtain the longer string length.
- Enter the longer and shorter string lengths to obtain the difference in cents.

The last section calculates the distances between frets of an instrument for a given temperament. For each row:

- Enter the two notes to be fretted on the same string and the sounding distance of the longer length, i.e. lower note (tangent to bridge).
- Enter the name of the worksheet for the temperament – this must be exact, so it is best to copy and paste the name.
- The frequencies of the two notes, the difference in cents, the length of the shorter string and the distance between the tangents are shown.

To use more rows, insert new rows before the last row. Likewise, remove rows by deleting the penultimate row, but do not delete rows 1 - 11 that contain the other calculators.

Of course this can also be used by measuring to the guide pins of the keys or markings for the tangent position on the keys.

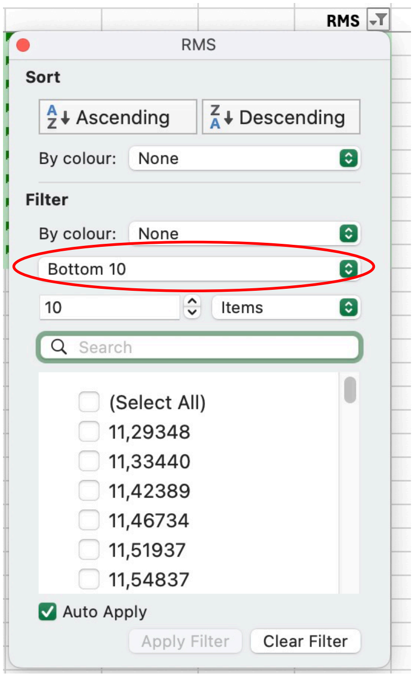
Theoretically, the difference in cents between the same two notes in different octaves should be the same, so this might be a plausibility check for the measurements.

3.2 Best Fit Fretting

This sheet tries to show which of the pre-defined temperaments is the best fit for an instrument fretting.

Put the name of an instrument sheet in the orange cell at the top left – it is best to copy and paste from the instrument worksheet name tab. The final column on the far right gives the RMS values of the frets with respect to the temperament and shows the minimum value at the top.

Note that “#N/A” values in the tables are normal if the instrument sheet does not fill the available space for frets. These are excluded from the RMS calculation.



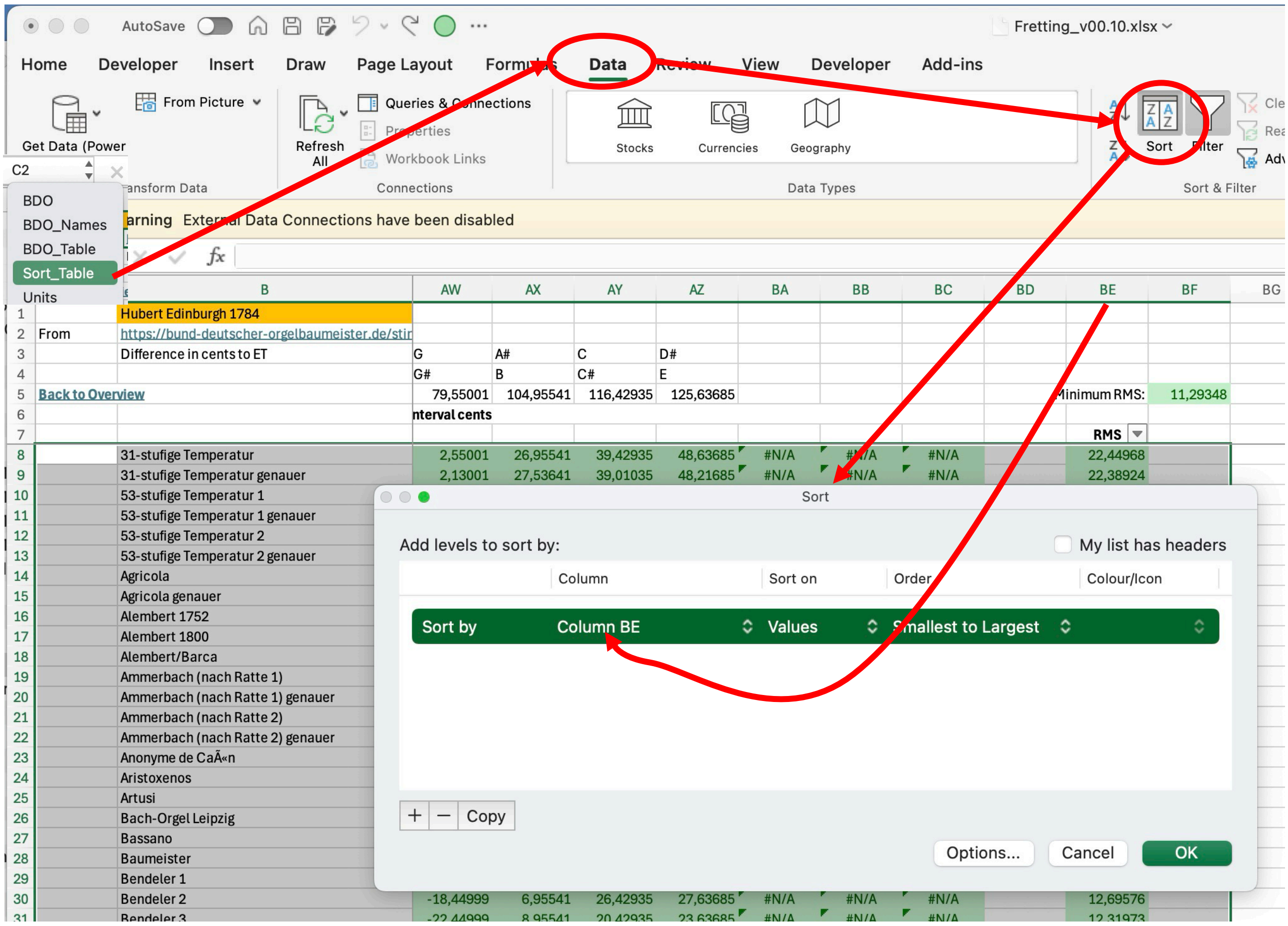
You can use the filter, for example to show the best 10 (Bottom 10 values) or values below a certain threshold.

Note that the filter does not sort the values, so you still have to find the minimum RMS value in the remaining list or sort the list as below.

The table can be sorted as follows and shown below:

- Unprotect the worksheet – see section 1.7.8.
- Select all the rows of the table by choosing "Sort_Table" from the name box at the top left.
- Use the menu Data – Sort to sort by the RMS column from smallest to largest.
- Re-instate the sheet protection.

Note: the sorting of the table will be reflected in the drop-down list in the 'BDO Any' worksheet, so it is best to sort the table alphabetically by column C before using this.



The result should be used with caution, especially with historical instruments, as apart from warping of the woodwork and repairs or modifications, they will probably have been retuned many times in their life, particularly by bending or repositioning the tangents.

3.3 Best Fit Temperament

This sheet shows which of the pre-defined temperaments is the best fit for a temperament sheet.

Enter the name of the temperament sheet in the orange cell at the top left and proceed as described for ‘Best Fit Fretting’ (section 3.2 above).

3.4 Best Fit Caveats

The best fit is based on the root mean square of the differences between the given temperament and the pre-defined temperaments.

1. For each note of the scale, the difference is squared to obtain the absolute value of the difference irrespective of whether it is lower or higher. Squaring also emphasises larger differences.
2. The differences for all the notes are added and divided by the number of notes to give an average or mean of the squared values.
3. The square root is taken.

Two temperaments with the same temperings on different notes, or temperings whose squares give the same sum will have the same RMS value, e.g. differences of 4, 5 and 6 have the same RMS as 5, 6 and 4.

Therefore the best fit may not be a similar temperament and the best fit must be compared note for note with the resulting best fit temperament.

4 Instrument Worksheets

4.1 Instrument Naming Conventions

The instrument sheets are named as follows:

Modern Maker	**Historical maker**	**Collection**	**Date**
Name of the modern maker of an original or copy, if applicable.	Name of the maker of the historical instrument. “anon.” if not known.	Place where currently located and registration number.	Year when made. ~ (tilde) for approximate.

Examples: Bavington Donat Leipzig 12 ~1700, Hubert Edinburgh 4338 1784.

Note that the names of tabs are limited to 31 characters, so abbreviations may be necessary.

The worksheet name is included in column titles and legends of the charts.

Further details should be provided in a text box within the sheet.

4.2 General Layout

A new instrument worksheet should be copied from the worksheet ‘Template Instrument Fretted’ (see section 6.1).

These are the main sheets to compare the fretting of a given instrument with various temperaments. Currently four different temperaments can be selected for comparison.

Fretted Notes				
Note 1	**Note 2**	**Longer**	**Shorter**	**Anon. Leipzig 10 ~1700 Fret Size cents**
F	F#	879,00000	851,00000	56,04484
G	G#	836,00000	811,00000	52,56123
C	C#	745,00000	711,00000	80,86904
D#	E	676,00000	656,00000	51,99292
F	F#	637,00000	615,00000	60,84835

The first section gives the fretting between Note 1 and Note 2 as the Longer and Shorter sounding string lengths; the distance between the frets is the difference longer minus shorter length. The units of length are arbitrary. The last column calculates the difference, i.e. the fret size, in cents.

Note: To use the correct frequency for C, it is sometimes necessary to enter “c oct.” instead of just “C”. This becomes apparent if the interval in the temperament is larger than 1000 cents.

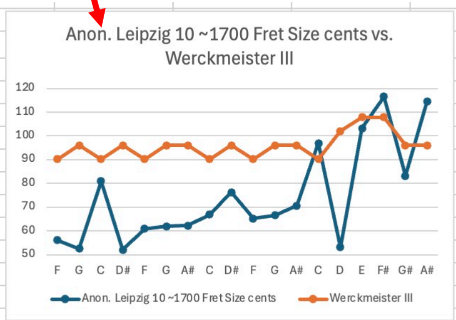
Meantone Oct 1			
Meantone Octave with Matching Octaves			
Meantone Oct 1 Frequency 1	**Meantone Oct 1 Frequency 2**	**Interval Cents**	**Difference**
470,79317	491,93496	76,04900	3,55129
263,18139	275,00000	76,04900	-4,78654
294,24573	307,45935	76,04900	-37,71763
307,45935	328,97673	117,10786	47,73361
352,00000	367,80717	76,04900	-13,22551
367,80717	393,54796	117,10786	2,84738
411,22091	440,00000	117,10786	5,37657
440,00000	470,79317	117,10786	7,45159

The other sections. look up the frequencies of Note 1 and Note 2 in the Worksheet named in the top, calculate the interval and the difference in cents in the last column.

The title from the referenced sheet is included as well, to show which temperament was selected when ‘BDO Any’ is used (see section 5.2).

Note: the referenced temperament worksheet names must correspond exactly with the names of the corresponding temperament worksheets themselves.

330,00	351,21	107,82	4,65
369,99	393,77	107,82	-8,75
416,24	440,00	96,09	12,97
468,27	495,00	96,09	-18,33
Anon., Leipzig 10, ~1700 Fret Size cents vs. Werckmeister III			
		RMS:	28,27



The merged cells at the bottom of the table form the title for the charts (Excel cannot currently do this directly in the title formula.)

Below this is the root mean square of the differences of the notes in cents. This gives an indication of the goodness of fit of the fretting to the given temperament. A lower number indicates a better fit. (This is generally used as a measure of how well an approximated function fits real data.) The formula is:

$$RMS = \sqrt{\frac{1}{n}\sum_{1}^{n} x_i^2}$$

The charts show the given fret sizes against each of the temperaments.

There is an instrument template for organ based on pipe lengths rather than frequencies – see section 6.1.2. The layout is similar to the above instrument sheets, but suitably simplified.

5 Temperament Worksheets

5.1 General Layout

A new temperament should be created by copying the appropriate temperament template (see 6.2).

We differentiate between theoretical temperaments, e.g. equal or meantone, and invented temperaments, e.g. Kirnberger III. There are some differences in the layouts of the theoretical sheets due to the way they are calculated.

A full title should be given at the top (cell B1) as the worksheet name has a limited length. This title is reproduced in the instrument sheets.

	Pythagorean Comma	Pythagorean Comma	1/12 Pythagorean Comma
Fraction 1/	12	0,986540369	0,998871385

The units to be used are defined. The drop-down list selects either Pythagorean or Syntonic comma or cents. Select the denominator of the fraction to be used in the cell below this, e.g. 4 for quarter units, 12 for twelfth units, 1 for whole units. The choice is reflected in in the neighbouring cells and is used in other column titles. The value for the whole unit and fractional unit are also shown for use in the calculations.

Note: to agree with the temperaments given in the literature, the correct unit and fraction must be used, especially the choice between Pythagorean and Syntonic comma.

Note: changing the units does **not** convert any data from one unit to another, it simply determines which units are used to interpret the given data.

A:	440	C for A=440:	261,6255653

To the right of this, the frequency for A can be entered in Hz and the frequency of C to achieve this is calculated.

The data entry then depends on the template used.

Tuning Defined in Fifths						
Note	Relative	Cumulative	F Hz	Cents btw. Fifths	Octaves down	F within octave
C	0	0	262,5133925		0	262,51339
G	-1	-1	392,88176	698,04500	0	392,88176
D	-1	-2	587,99315	698,04500	1	293,99658
A	-1	-3	880,00000	698,04500	1	440,00000

For a tuning defined in fifths, the tempering of the fifths is entered in units depending on the choice above, e.g. quarter Pythagorean commas. The other columns are calculated. It may be necessary to adjust the "Octaves down" to keep the "Frequency within octave" between C and c oct.

Tuning Defined in Cents from C				
Note Scale Order	Cents from C	F Scale Order	Relative Frequency Cents	Monochord String Length
C	0,00000	262,51339		3600,00
C#	86,31499	275,93342	86,315	3424,91
D	196,089998	293,99658	109,775	3214,49
D#	305,865003	313,24219	109,775	3016,99

For a tuning defined as a chromatic scale in cents, the values are entered in that order.

Tuning Defined in Cents Relative to ET				
Note Scale Order	ET Frequency Hz	Difference to ET Scale in Cents	Resulting F	Monochord String Length
C	262,51339	0,00000	262,51339	3600,00
C#	278,12325	-13,68501	275,93342	3424,91
D	294,66132	-3,91000	293,99658	3214,49
D#	312,18279	5,86500	313,24219	3016,99

For a tuning defined in cents relative to equal temperament, the procedure is similar. The ET frequencies are given and the differences entered to give the resulting frequencies and string lengths.

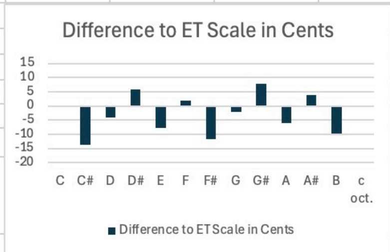
The next section either converts the fifths to scale order or vice-versa,depending on the template used.

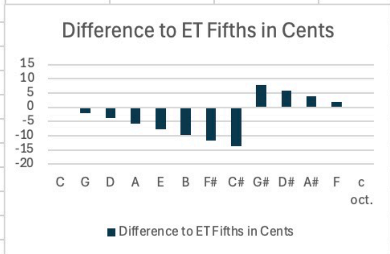
Monochord String Length
3600,00
3424,91
3214,49

The string length for a monochord string is also show. The basis for this can be entered. This can also be used to show the lengths of organ pipes.

The section for equal temperament calculates the equal temperement notes and the differences to the current tuning, both in scale order and fifths order. These are both visualised in the accompanying graphics. The “ET in Fifths” and “Difference to ET Fifths in Cents” columns are named for use with the ‘Best Fit Temperament’ worksheet.

Compared to Equal Temperament					
ET in scale order	ET Frequency Hz	Difference to ET Scale in Cents	ET in Fifths	Frequency Hz	Difference to ET Fifths in Cents
C	262,51339	0,000	C	262,51339	0,000
C#	278,12325	-13,685	G	392,88176	-1,955
D	294,66132	-3,910	D	293,99658	-3,910
D#	312,18279	5,865	A	440,00000	-5,865
E	330,74615	-7,820	E	329,25553	-7,820
F	350,41334	1,955	B	492,76912	-9,775
F#	371,25000	-11,730	F#	368,74309	-11,730
G	393,32567	-1,955	C#	275,93342	-13,685
G#	416,71404	7,820	G#	418,60059	7,820
A	441,49314	-5,865	D#	313,24219	5,865
A#	467,74569	3,910	A#	468,80329	3,910
B	495,55930	-9,775	F	350,80927	1,955
c oct.	525,02679	0,000	c oct.	525,02679	0,000





Note: since the notes are given starting from C, the A in Equal Temperament is not the value selected, e.g. 440Hz.

There is a section to compare the major thirds to the just interval, both in cents and in the units chosen at the top of the sheet.The just interval in cents is given at the top for comparison with the values of the temperament.

Thirds Compared to Just			386,31	Cents	
Lower Note	Major Third	Octave Adjustment	Kirnberger col. 1 Interval in cents	F1 *5/4	diff 1/1 Cents
C	E	1	386,31	328,14	0,00
C#	F	1	407,82	345,70	21,49
D	F#	1	386,31	369,16	0,00
D#	G	1	407,82	388,91	21,49
E	G#	1	405,87	410,18	19,54
F	A	1	396,09	437,52	9,77
F#	A#	1	405,87	461,45	19,54
G	B	1	386,31	492,21	0,00
G#	c oct.	1	407,82	518,54	21,49
A	C#	2	396,09	550,00	9,77
A#	D	2	407,82	583,36	21,49
B	D#	2	405,87	615,27	19,54

Beside this is a section to show the fifths and thirds as ratios (1:x). This is useful to see how pure the fifths and thirds are (compared to 1.5 and 1.25 respectively). You may wish to show more decimal places to see finer differences – see section 1.7.5.

Ratios						
Lower Note	Fifth	Octave Adjustment	Fifths Ratio	Lower Note	Major Third	Thirds Ratio
C	G	1	1,50	C	E	1,25
C#	G#	1	1,52	C#	F	1,27
D	A	1	1,50	D	F#	1,25
D#	A#	1	1,50	D#	G	1,25
E	B	1	1,50	E	G#	1,27
F	c oct.	1	1,50	F	A	1,25
F#	C#	2	1,50	F#	A#	1,27
G	D	2	1,50	G	B	1,25
G#	D#	2	1,50	G#	c oct.	1,25
A	E	2	1,50	A	C#	1,25
A#	F	2	1,50	A#	D	1,25
B	F#	2	1,50	B	D#	1,27

The harmonics section (not shown) is an additon to investigate the harmonics of the note C. They are calculated by doubling the intial frequency and then bringing them down to the middle octave. The nearest note in the temperament is found and the difference isshown. The nearest note can have a very different frequency to the harmonic so as not to be truly considered as such – compare the frequency of the harmonic with that of the given nearest note in the Master Frequencies table..

Lastly there is an octave transposer to find the frequency of any of the notes in any octave above or below the middle octave.

The “Master Frequencies” table is present for technical reasons – see Appendix B.4.

5.2 BDO Any

The ‘BDO Any’ worksheet allows you to select any temperament from the [BDO] data from the drop-down list. (This is the same data as in the ‘Best Fit Fretting’ worksheet – see 3.2. Note that the list will be shown in the order in which it was last sorted there.)

The units should not be changed from single cents, as this is the format of the BDO data.

This worksheet can be referenced from Instrument worksheets. It can be copied and renamed to provide a selection of temperaments instead of creating them from the templates. They can then be used for comparison with an instrument. The name of the selected BDO temperament is reproduced in the instrument sheet (see section 4.2).

5.3 Equal Temperament

This is directly calculated by dividing the octave into 12 equal semitones of 100 cents each using the approprate template..

The comparison with Equal Temperament (i.e. compared with itself) on the right with the graphs is included for compatibility with the other temperaments.

5.4 Kirnberger

According to [Bellermann, 1871], the first part of [Kirnberger, 1771-79] was initially published in 1771 and then reprinted with different titles and introductions until 1774, as the sales did not initially go well. In the last letter to Forkel published in [Bellermann, 1871] (the last page with the date was missing) Kirnberger gives the following temperaments:

Table on page 572 of [Bellermann, 1871]:

Interval	1/12 commas	1/12 commas	1/12 commas
C – G			-2
G – D			-2.5
D – A	-6	5 ½	-3.5
A – E	-5	5 ½	-3
F# - C#	-1	-1	-1

Bei der ersten Art, in welcher *Fis-cis* gegen $^{2}/_{3}$ um $^{1}/_{12}$ des Quinten-Excesses zu tief ist, und *D-A* und *A-e* beide zusammen um $^{11}/_{12}$, so dass *D-A* $5^{1}/_{2}$ und *A-e* auch $5^{1}/_{2}$ hat, oder wenn man will, *D-A* $^{6}/_{12}$ oder $^{1}/_{2}$ Comma von $^{80}/_{81}$ und *A-e* $^{5}/_{12}$

The text describes the first two columns above in reverse order:

The third column is not mentioned.

In [Bellermann, 1871] Kirnberger gives the following method to divide the Syntonic comma over four fifths:

Oder wenn man von *C* nach *e* 80 : 81 in vier Quinten vertheilen will, kann es folgender Art geschehen:

C-G **216 : 323 temperirte Quinte = $^{2}/_{3}$ — $^{1}/_{324}$**
216 : 324 reine Quinte

G-d **$215^{1}/_{3}$: 322 temperirte Quinte = $^{2}/_{3}$ — $^{1}/_{323}$**
$215^{1}/_{3}$: 323 reine Quinte

A-e **$214^{2}/_{3}$: 321 temperirte Quinte = $^{2}/_{3}$ — $^{1}/_{322}$**
$214^{2}/_{3}$: 322 reine Quinte

D-A **214 : 320 temperirte Quinte = $^{2}/_{3}$ — $^{1}/_{321}$.**
214 : 321 reine Quinte.

These ratios give:

Interval			cents
C – G	323	324	-5,3515778
G - d	322	323	-5,3681719
A - e	321	322	-5,3848692
D - A	320	321	-5,4016707
Average			-5,3765724

The average is exactly one quarter of a Syntonic comma, 80:81.

This seems to be the temperament commonly regarded as "Kirnberger III", although it is not included in his table of three temperaments on page 572 of [Bellermann, 1871] above.

However when working in Syntonic commas, these do not give a pure octave for C, they need an adjustment of one schisma or -0.36338 quarter Syntonic commas; the schisma is by definition the difference between the Pathagorean and Syntonic comma. Most sources (e.g. Wikipedia and [BDO] "Kirnberger 3 (nach[2] Lange) genauer[3]") put it between F# - C#.

Note	Relative 1/4 Comma	Cumulative 1/4 Comma	F Hz	Cents btw. Fifths	Octaves down	F within octave	Note Scale Order	F Scale Order	Cents from C
C	0	0	263,1813855		0	263,18139	C	263,18139	0,000
G	-1	-1	393,54796	696,58	0	393,54796	C#	277,26105	90,225
D	-1	-2	588,49147	696,58	1	294,24573	D	294,24573	193,157
A	-1	-3	880,00000	696,58	1	440,00000	D#	311,91868	294,135
E	-1	-4	1315,90693	696,58	2	328,97673	E	328,97673	386,314
B	0	-4	1973,86039	701,96	2	493,46510	F	350,90851	498,045
F#	0	-4	2960,79059	701,96	3	370,09882	F#	370,09882	590,224
C#	-0,36338	-4,363376635	4436,17677	700,00	4	277,26105	G	393,54796	696,578
G#	0	-4,363376635	6654,26515	701,96	4	415,89157	G#	415,89157	792,180
D#	0	-4,363376635	9981,39773	701,96	5	311,91868	A	440,00000	889,735
A#	0	-4,363376635	14972,09660	701,96	5	467,87802	A#	467,87802	996,090
F	0	-4,363376635	22458,14490	701,96	6	350,90851	B	493,46510	1088,269
c oct.	0	-4,363376635	33687,21734	701,96	6	526,36277	c oct.	526,36277	1200,000

With Pythagorean quarter commas it works out without adjustment:

In the letters [Bellermann, 1871], Kirnberger calls the comma "Quinten-Exzess" which we would interpret as Pythagorean commas, i.e. the left over on the octave after completing a cycle of fifths, however this conflicts with his sudden mention of 80:81 in the above text.

Note	Relative 1/4 Comma	Cumulative 1/4 Comma	F Hz	Cents btw. Fifths	Octaves down	F within octave	Note Scale Order	F Scale Order	Cents from C
C	0	0	263,4042326		0	263,40423	C	263,40423	0,000
G	-1	-1	393,77009	696,09	0	393,77009	C#	277,49582	90,225
D	-1	-2	588,65752	696,09	1	294,32876	D	294,32876	192,180
A	-1	-3	880,00000	696,09	1	440,00000	D#	312,18279	294,135
E	-1	-4	1315,53573	696,09	2	328,88393	E	328,88393	384,360
B	0	-4	1973,30359	701,96	2	493,32590	F	351,20564	498,045
F#	0	-4	2959,95538	701,96	3	369,99442	F#	369,99442	588,270
C#	0	-4	4439,93307	701,96	4	277,49582	G	393,77009	696,090
G#	0	-4	6659,89961	701,96	4	416,24373	G#	416,24373	792,180
D#	0	-4	9989,84941	701,96	5	312,18279	A	440,00000	888,270
A#	0	-4	14984,77412	701,96	5	468,27419	A#	468,27419	996,090
F	0	-4	22477,16118	701,96	6	351,20564	B	493,32590	1086,315
c oct.	0	-4	33715,74177	701,96	6	526,80847	c oct.	526,80847	1200,000

All these temperaments are given in the following sheets.

---

[2] "after"

[3] "more exact"

5.4.1 Kirnberger col. 1

Column 1 of Kirnberger's table in [Bellermann, 1871] page 572.

5.4.2 Kirnberger col. 2

Column 2 of Kirnberger's table in [Bellermann, 1871] page 572.

5.4.3 Kirnberger col. 3

Column 3 of Kirnberger's table in [Bellermann, 1871] page 572.

5.4.4 Kirnberger III S.

The ratios given by Kirnberger in in [Bellermann, 1871] page 571 using Syntonic commas. This has an extra schisma, calculated in ¼ commas in cells M1 – O3:

The frequency *f* of C# raised by one Schisma *s: C#'* = $f2^{\frac{s}{1200}}$

Difference between this and original C# in quarter commas *q*: Δ= $log_q \frac{C\#'}{C\#}$

5.4.5 Kirnberger III P.

The ratios given by Kirnberger in [Bellermann, 1871] page 571 using Pythagorean commas.

5.5 Meantone Octave 1 and 2

The 'Meantone Oct' sheets are versions of a meantone temperament that keeps the octaves exact.

They use the Syntonic comma.

Meantone Octave 1 is taken from [Benson, 2007]. This differentiates between D# and Eb and calculates the last three notes downwards from the c octave.

With Meantone Octave 2, omitting Eb, all fifths are gven -1 quarter comma, and then the c-octave must have +6,636623 quarter commas to reach the octave. With the -1 quarter comma that would normally be used this is 7.636623, which is the three syntonic commas minus a Pythagorean comma 128:125:

$$log_q\left(\frac{128}{125}\right) = 7.636623$$

Where q is the quarter syntonic comma (see [Benson, 2007] p. 186).

5.6 Meantone Quarter Comma

The 'Meantone q. comma' sheet reduces all fifths by a quarter comma.

The octaves of C and G do not match.

5.7 Meantone Bavington Donat

'Meantone Bav Donat' is the first tentative conclusion by [Bavington, 2002] of a possible temperament for the Donat instrument (Leipzig 12):

"a kind of modified quarter-comma meantone, with a very low C# (for reasons unknown) and with Eb and perhaps Bb given a kind of compromise tuning enabling them to serve for D# and A#"

The best 10 fits are:

	Minimum RMS:	2,78319
	RMS	
Lambert-Chaumont	2,78319	
Rameau (nach Mercadier/Legros)	2,91931	
Rameau (nach Mercadier/Barbieri)	3,22875	
Sieur Vincent	4,02483	
Alembert 1752	4,25691	
Rameau (nach Riche)	4,55957	
Corette	4,57623	
Anonyme de Caën	4,78954	
mitteltönig modifiziert (nach Vogel 2)	4,93952	
Legros 2	5,04669	

The Lambert-Chaumont is described by [BDO] as:

"Meantone tuning with pure major thirds between C-E and A-C#. Because the major thirds above B and Eb are too wide by 22 and 20 cents respectively, the Eb can also be used as D#. The widest major thirds are +41 cents above C# and G#. The wolf fifth si +16 cents above G#."

5.8 Monochord

This sheet is included to reproduce the original tuning from [Werckmeister, 1691].

The first section gives the calculated lengths and frequencies for the theoretical intervals from C calculated by proportions as in [Werckmeister, 1691] p.39 (see Figure 5-2).

[Werckmeister, 1691] p. 77 gives table for 120 and 10800 as the starting length for C, with the first stated as easier to measure with compasses on a 3 or 4 foot monochord (see Figure 5-1). He does not give the units, but any can be used, e.g. millimetres, centimetres or inches.

The sounding length is based on C = 120 (set in the first cell). The length units are arbitrary, e.g. cm. These match:

- the upper table on Werckmeister p.77 when C is set to a length of 120
- the corrected lower table when C is set to 10800
- the lengths on the "Kupferstich"

The corresponding frequencies are based directly on A= 440Hz (set within the table).

The other sections correspond to the templates (see above).

C	CIS	25-24 Semitonium minus.
	Cisd.	16-15 Semitonium majus.
	D.m.	10-9 Tonus minor,
	D	9-8 Tonus major.
	Dis	75-64 Tertia minor diesi Enharmonica deficiens.
	E.m.	6-5 Tertia minor.
	E	5-4 Tertia major.
	F	4-3 Quarta.
	Fis m	25-18 quarta Semitonio minori abundans.
	Fis	45-32 quarta semitonio min. & commate abundans.
C	G	3-2 quinta.
	Gis	25-16 quinta Semitonio minori abundans.
	A m.	8-5 Sexta minor.
	A	5-3 Sexta major.
	A. d.	27-16 Sexta major commate abundans.
	A. s.	225-128 Sexta major Semit min. & commate abundans.
	B m.	16-9 Septima minor commate deficiens.
	B.	9-5 Septima minor.
	H	15-8 Septima major.
C	c	2-1 Octava.

Figure 5-2Werckmeister Musicalische Temperatur p. 39.

c	cis	cis dur.	dm.	d.	dis.	em.	e.	f.
120.	$115\frac{1}{5}$.	$112\frac{1}{2}$.	108.	$106\frac{2}{3}$.	$102\frac{2}{5}$.	100.	96.	90.

fis m.	fis.	g.	gis.	a m.	a.	a dur.	as.	b m.	b.
$86\frac{2}{3}$.	$85\frac{1}{3}$.	80.	$76\frac{4}{5}$.	75.	72.	$71\frac{1}{9}$.	$68\frac{4}{15}$.	$67\frac{1}{2}$.	$66\frac{2}{3}$.

h.	c.
64.	60.

Diese Zahlen sind so niedrig genommen / weil man sie mit dem Zirckel auff ein Monochordum von 3. oder 4. Fuß eintheilen und aufftragen kan/welches in den nachfolgenden grossen Zahlen nicht geschehen kan / ist auch einerley / nur daß man der Brüche Umgang haben kan.

C	Cis	Cis	Cis dur.	Dm.	Dis	Dis.	Es.
10800.	10368.	10125.	9720.	9600.	9216.	9000.	8640.

E.	F.	Fis m.	Fis.	G.	Gis.	A m..	A.	A dur.
8640.	8100.	7776.	7680.	7200.	6912.	6750.	6480.	6400.

AS.	B m.	B.	H.	c.
6144.	6075.	6000.	5760.	5400.

Figure 5-1 Werkmeister Musicalische Temperatur p. 77

The results in the spreadsheet agree with Werckmeister's lengths and in his "Kupferstich" diagram (Figure 5-3).

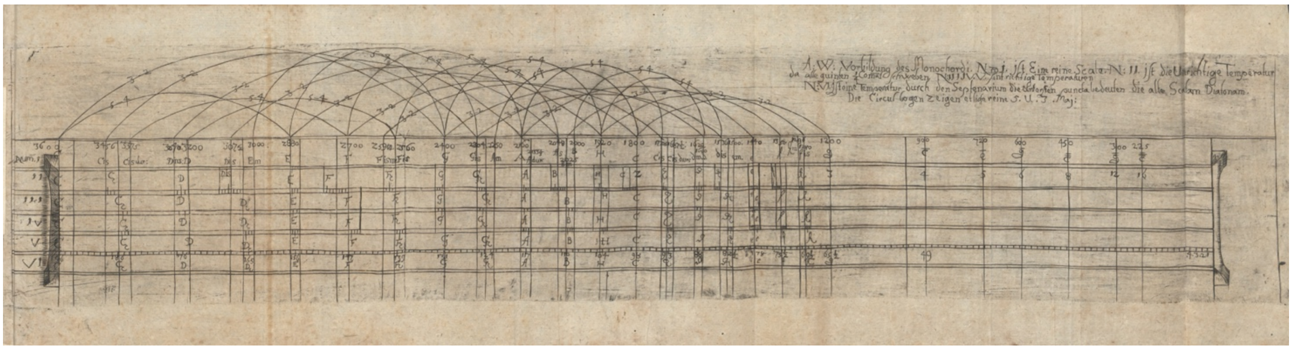


Figure 5-3 Werckmeister's "Kupferstich" Engraving

5.9 Neidhardt

According to [Norrback,2002] Neidhardt changed the names of his temperaments for courts and different sized towns between 1724 and 1732. Here we give both versions.

5.10 Neidhardt 1

[Neidhardt, 1724] p.12: "für ein Dorf", [Neidhardt, 1732]: "für eine kleine Stadt".

[Neidhardt, 1724] p.12 gives the fifths reduced by twelfths of commas (reading upwards from the bottom of his table).

Setting the monochord string length to 2000 gives the results of [Neidhardt, 1724] p. 16, if we assume that Neidhardt truncated the decimals instead of rounding them.

This agrees with [Benson, 2007] p. 193.

The major thirds all come out as one cent less than Neidhardt. According to [Lehmann, 2006] this is because Neidhardt uses Syntonic commas at this point.

Comparing the results for the major thirds of Neidhardt 1 compared to the pure 5:4 interval in twelfth Syntonic commas, C-E and G-B still give one less than Neidhardt. Is this another rounding problem of Neidhardt's?

Note	Third	Diff. 1/12 commas	Rounded	Neidhardt
C	E	-3,27325	-3	4
C#	F	-9,81831	-10	10
D	F#	-5,45494	-6	6
D#	G	-8,72747	-9	9
E	G#	-9,81831	-10	10
F	A	-5,45494	-6	6
F#	A#	-9,81831	-10	10
G	B	-4,36409	-4	5
G#	c oct.	-9,81831	-10	10
A	C#	-7,63662	-8	8
A#	d	-7,63662	-8	8
B	D#	-9,81831	-10	10

5.11 Neidhardt 2

[Neidhardt, 1724] p.13: “für eine kleine Stadt”, [Neidhardt, 1732]: “für eine große Stadt”.

[Neidhardt, 1724] p.13 gives the fifths reduced by twelfths of commas (reading upwards from the bottom of his table).

Setting the monochord string length to 2000 gives the results of [Neidhardt, 1724] p. 17 with some small differences in the second decimal place, if we assume that Neidhardt truncated the decimals instead of rounding them.

This agrees with [Benson, 2007] p. 193 except that he gives E differently as -2/3 as the fifth from A and -7/12 as the major third from C.

5.12 Neidhardt 3

[Neidhardt, 1724] p.14: “für eine große Stadt”

[Neidhardt, 1724] p. 14 gives the fifths reduced by twelfths of commas (reading upwards from the bottom of his table).

This does not agree with [Neidhardt, 1724] p.18., only D and G agree well:

Note Scale Order	Monochord String Length	[Neidhardt, 1724] p.18
C	2000,00	2000,00
C#	1889,88	1892,01
D	1785,83	1785,82
D#	1681,79	1683,68
E	1589,19	1592,78
F	1496,62	1500,00
F#	1415,81	1417,40
G	1336,35	1336,34
G#	1259,92	1262,76
A	1191,90	1193,23
A#	1122,46	1123,72
B	1060,66	1061,85
c oct.	1000,00	1000,00

5.13 Pythagorean

The ‘Pythagorean’ sheet shows the values for the Pythagorean scale with all the fifths in the ratio of 3:2 or a factor of 1.5. These are calculated directly in the Fifths section.

Note that the octave c oct. is not twice the frequency of C, the difference being the Pythagorean comma. This is shown below the “Scale Order from C” table compared with the fractional form of the Pythagorean comma and the Syntonic comma 81/80.

5.14 Silbermann Freiberg 1 - 4

The first two sheets are values given in [Wegscheider/Schütz 1988] p.78 and 79 for the measured lengths of some original pipes on the Silbermann organ in the Freiberg cathedral. ‘Silbermann

Freiberg 3' is the final tuning after the restoration in September 1983 as described on p. 86 (with the final modifications decided on 22.06.1983).

'Silbermann Freiberg 4' is the re-tuning of September 1985 according to the suggestion by Christoph Schwarzenberg of 1982.

They are mainly included to show the results of the Best Fit: the first exactly matches "Silbermann, Gottfried (nach Wegscheider 3)" and the second exactly matches "Silbermann, Gottfried (nacg Wegscheider 4)". These confirm that the worksheets are correct – both BDO datasets give sources from the organ builders Jehmlich of 1982, for whom Wegscheider apparently worked, which are also referenced by [Wegscheider/Schütz 1988]. The third matches "Silbermann, Gottfried (nach Wegscheider 5)" very closely. 'Silbermann Freiberg 4' matches "Freiberg Dom 1985 genauer[4]" very closely.

Note: the sheets use the 'Template from Scale' and 'Template from Fifths' and not 'Template Instrument Organ' as the values are given in cents from C in scale order.

5.15 Silbermann-Sorge and Silbermann-Wegscheider 1

These are from [Norrback,2002] p. 81.

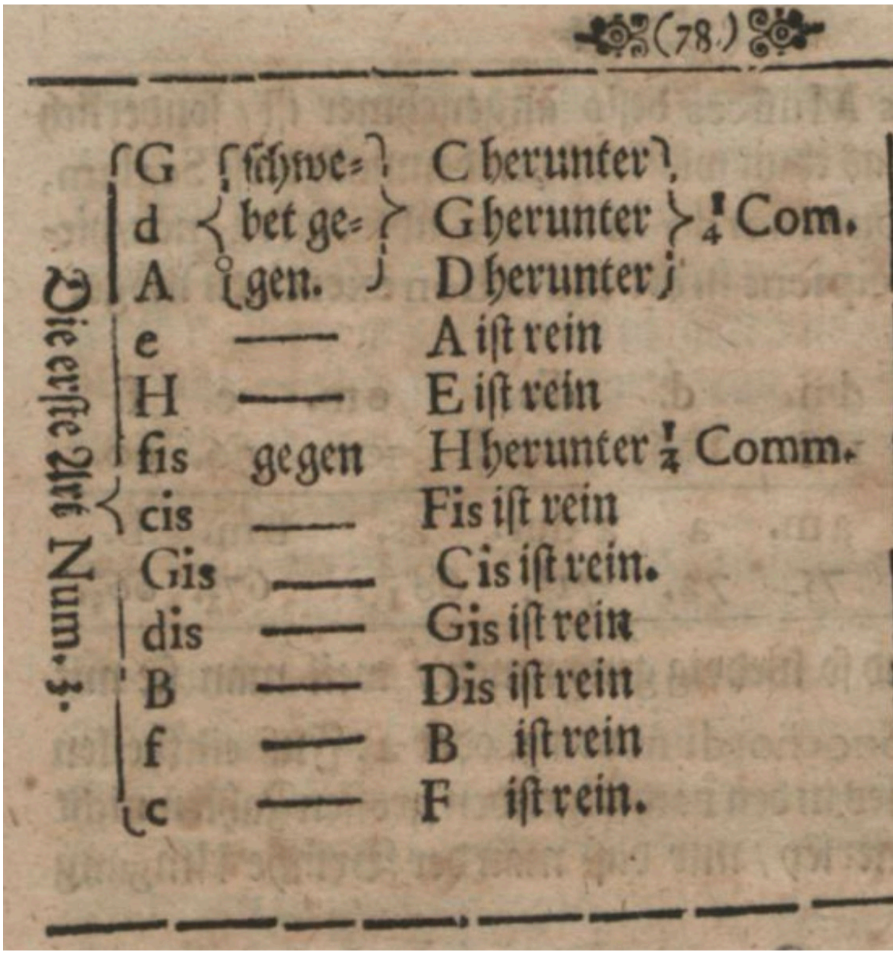
5.16 Valotti-Young

This is from [Benson, 2007] and agrees closely with [BDO] "Vallotti genauer".

[4] "more exactly"

5.17 Werckmeister III

This sheet derives the values of the temperament from [Werckmeister, 1691]. The relative quarter comma tuning of each fifth is taken from there (p.78) and agrees with [Benson, 2007], p. 190 and other places.


(78)

Die erſte Art Num. 3.

G	ſchwe-	C herunter	
d	bet ge-	G herunter	$\frac{1}{4}$ Com.
A	gen.	D herunter	
e	—	A iſt rein	
H	—	E iſt rein	
fis	gegen	H herunter	$\frac{1}{4}$ Comm.
cis	—	Fis iſt rein	
Gis	—	Cis iſt rein.	
dis	—	Gis iſt rein	
B	—	Dis iſt rein	
f	—	B iſt rein	
c	—	F iſt rein.	

Figure 5-4 Werckmeister P.78

6 Template Worksheets

6.1 Instrument Templates

6.1.1 Template Instrument Fretted

Copy the ‘Template Instrument Fretted’ worksheet and rename the sheet (see 1.7.6).

Change the worksheet colour to the colour for instruments (see 1.7.7).

Add it to the indexes in the Overview sheet (see 2.1).

Enter the full name in cell B1.

Enter the data in the first section. To make more rows insert rows before the last row to ensure the graphs include them all. To remove rows, delete any entire rows. (The worksheet must be unprotected to do this – see section 1.7.8.

To compare with temperaments, type the names of the desired temperaments in cells at the top left of the tables. These must exactly match the sheet names, so it may be easier to use copy and paste them. The ‘BDO Any’ worksheet can also be referenced here (with the new name if renamed), and the result will be shown for the temperaments selected there.

6.1.2 Template Instrument Organ

This is used to define the temperament of an instrument using lengths rather than frequencies. These can be pipe or string lengths.

Copy the ‘Template Instrument Organ’and rename the sheet (see 1.7.6).

Change the worksheet colour to the colour for instruments (see 1.7.7).

Add it to the indexes in the Overview sheet (see 2.1).

Enter the full name in cell B1.

Enter the pipe lengths of one octave. The layout is similar to the instrument sheets, but suitably simplified and with the addition of the comparison with equal temperament.

> Note: to find the best fitting temperament use the ‘Best Fit Temperament’ worksheet, not the ‘Best Fit Fretting’ worksheet.

6.2 Temperament Templates

6.2.1 Template from Fifths

This is used to define a temperament in a cycle of fifths giving the deviations from the just interval.

The units can be cents, Pythagorean commas, Syntonic commas, or fractions thereof.

The frequency of A can be defined.

6.2.2 Template from Scale

This is used to define a temperament with the notes in scale order (C – C# - D - …) with the notes defined relative to the starting C.

The units can be cents, Pythagorean commas, Syntonic commas, or fractions thereof.

The frequency of A can be defined.

6.2.3 Template from ET Fifths Order

This is used to define a temperament in a cycle of fifths giving the deviations from note on the equal tempered scale.

The units can be cents, Pythagorean commas, Syntonic commas, or fractions thereof.

The frequency of A can be defined.

6.2.4 Template from ET Scale Order

This is used to define a temperament with the notes in scale order (C – C# - D - …) giving the deviations from note on the equal tempered scale.

The units can be cents, Pythagorean commas, Syntonic commas, or fractions thereof.

The frequency of A can be defined.

6.3 Starting from A

If you wish to start the tuning from A rather than C, a temperament sheet or template must be modified as follows. Note: it is best to use Paste Values rather than Paste to keep the formatting and borders.

1. In the “Tuning Defined in Fifths” section:
    a. change the order of the “Note” and “Relative” columns to list the fifths starting with A instead of C. The last entry should be “a oct.” with the appropriate temperament.
    b. Replace the formula for the first entry in the “F Hz” column to refer to the desired frequency of A rather than the calculated frequency of C, i.e. from “=K2” to “=I2”.
2. In the “Master Frequencies” section: change the order in the “Note Scale Order” column to start at A up to a oct.
3. In the “Scale Order from C” section:
    a. change the title to “Scale Order from A”
    b. change the title of the column “Cents from C” to “Cents from A”.
    c. copy and paste the “Note Scale Order” column from the Master Frequencies section to give the scale order from A.
4. In the “Thirds Compared to Just” section:
    a. In the “Major Third” column change the “c oct.” to “C”
    b. In the “Octave Adjustment” column, change the values to keep within the octave, i.e. change F-A, F#-A#, G-B and G#-C from 1 to 2 and change A-C#, A#-D and B-D# from 2 to 1.
5. In the “Ratios” section:
    a. in the “Fifth” and “Major Third” columns, change “c oct.” to “C”.
    b. In the “Fifths Ratio” column, add or remove “2*” to the formula.
6. In the “Octave Transposer” section: replace the “Note” column with the list starting from A.

7 Bibliography

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[NetCat]	NetCat Inc. Instrument Tuner http://www.instrument-tuner.com/temperaments.html
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[Vermeij, 2000]	Vermeij, Koen. The Hubert Clavichord Data Book: a description of all extant clavichords by Christian Gottlob Hubert 1714-1793. Clavichord International Press, 2000
[Werckmeister, 1691]	Werckmeister, Andreas. Musicalische Temperatur; Quedlinburg 1691.

Appendix A Formulae

This appendix lists the essential formulae. See also B.4.5 Calculation of C for a Given A.

Cents is a logarithmic difference between two notes obtained by dividing the octave of 12 notes into 1200 steps. The difference in cents between two notes is the same in all octaves.

For frequencies f_1 and f_2 and cents represented by ¢

$$Interval\ in\ ¢ = 1200 * log_2\left(\frac{f_2}{f_1}\right)$$

An equal tempered semitone is 100¢.

$$f_2 = f_1 2^{¢/1200}$$

$$Pythagorean\ comma = \frac{3^{12}}{2^{19}} = \frac{531441}{524288}$$

$$Syntonic\ comma = \frac{81/64}{5/4} = \frac{81}{80}$$

Schisma (difference between the Pythagorean and Syntonic commas):

$$Schisma = \frac{531441/524288}{81/80} = \frac{32805}{32768}$$

$$Schisma\ in\ commas = \frac{Schisma\ in\ cents}{Comma\ in\ cents}$$

$$Schisma\ in \frac{1}{4} commas = 4 * Schisma\ in\ commas = 4 * \left(\frac{Schisma\ in\ cents}{Comma\ in\ cents}\right) = \frac{Schisma\ in\ cents}{\frac{1}{4} comma\ in\ cents}$$

Fraction of comma, e.g. quarter comma

$$q = comma^{1/4}$$

Interval x in fractions of comma q:

$$x = \ log_q \frac{f2}{f1}$$

$$f_2 = f_1 q^x$$

Quarter comma in cents:

$$¢ = 1200 * log_2(q) = 1200 * log_2\left(comma^{\frac{1}{4}}\right) = \frac{1}{4} * 1200 * log_2(comma)$$

A quarter comma is a comma to the power of a quarter, i.e. $x^{1/4}$ when working in fractions or real numbers but is a comma/4 when working in cents:

Frequency is inversely proportional to length so that

cents F2/F1 = cents L1/L2 or

cents F2/F1= -cents L2/L1.

Appendix B Excel Implementation

This appendix gives some technical details of the implementation in Excel.

B.1 Colours

The tabs of the worksheets are coloured as follows (see also 1.7.7):

Sheets	**Excel Colour**
Overview and General InformationSheets	“White, Background 1, Darker 15%”
General Calculations	“Dark blue, Text 2, Lighter 75%”
Instrument Sheets	“Green, Accent 6, Lighter 60%”
Theoretical Temperaments	“Orange, Accent 2, Lighter 60%”
Invented Temperaments	“Plum, Accent 5, Lighter 60%”
Variable Temperaments	“Orange Accent 2, Darker 25%”
Templates	“Yellow”

B.2 Protection

Each sheet has a conditional formatting to show unlocked cells with an orange background and cells with a formula with a light green background.

Allowing changes, e.g. Sort or Insert rows or Delete rows does not work if individual cells are protected, so these functions cannot be performed without unprotecting the worksheet.

B.3 Instrument Worksheets

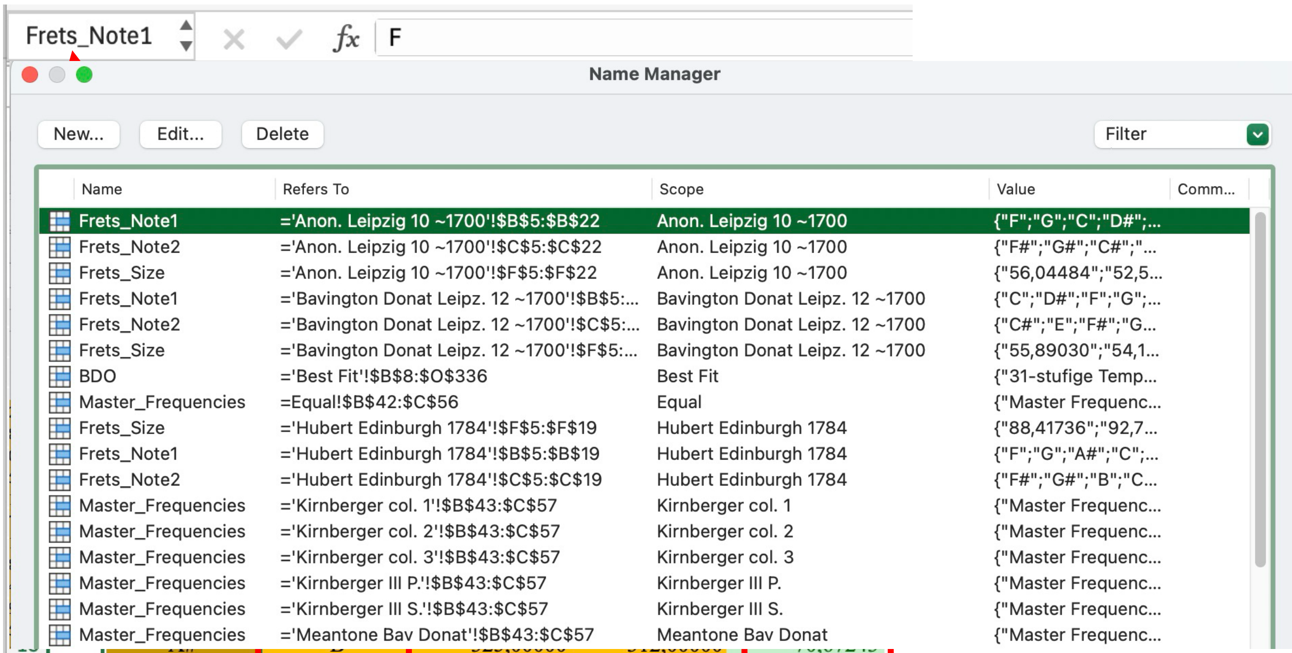
See 4.2 for the basic overview.

B.3.1 Named Cell Ranges

The cells for Note 1, Note 2 and the fret sizes must be named as shown in Figure_Apx B-1

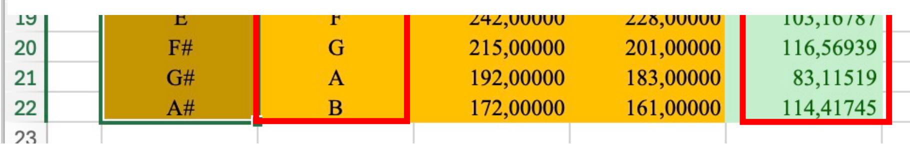
- “Frets_Note1” The Note 1 data
- “Frets_Note2” The Note 2 data
- “Frets_Size” The Fret Size data

Note that the scope of the name must be defined for the worksheet and not for the entire workbook.



Name	Refers To	Scope	Value	Comm...
Frets_Note1	='Anon. Leipzig 10 ~1700'!$B$5:$B$22	Anon. Leipzig 10 ~1700	{"F";"G";"C";"D#";...	
Frets_Note2	='Anon. Leipzig 10 ~1700'!$C$5:$C$22	Anon. Leipzig 10 ~1700	{"F#";"G#";"C#";"...	
Frets_Size	='Anon. Leipzig 10 ~1700'!$F$5:$F$22	Anon. Leipzig 10 ~1700	{"56,04484";"52,5...	
Frets_Note1	='Bavington Donat Leipz. 12 ~1700'!$B$5:...	Bavington Donat Leipz. 12 ~1700	{"C";"D#";"F";"G";...	
Frets_Note2	='Bavington Donat Leipz. 12 ~1700'!$C$5:...	Bavington Donat Leipz. 12 ~1700	{"C#";"E";"F#";"G...	
Frets_Size	='Bavington Donat Leipz. 12 ~1700'!$F$5:...	Bavington Donat Leipz. 12 ~1700	{"55,89030";"54,1...	
BDO	='Best Fit'!$B$8:$O$336	Best Fit	{"31-stufige Temp...	
Master_Frequencies	=Equal!$B$42:$C$56	Equal	{"Master Frequenc...	
Frets_Size	='Hubert Edinburgh 1784'!$F$5:$F$19	Hubert Edinburgh 1784	{"88,41736";"92,7...	
Frets_Note1	='Hubert Edinburgh 1784'!$B$5:$B$19	Hubert Edinburgh 1784	{"F";"G";"A#";"C";...	
Frets_Note2	='Hubert Edinburgh 1784'!$C$5:$C$19	Hubert Edinburgh 1784	{"F#";"G#";"B";"C...	
Master_Frequencies	='Kirnberger col. 1'!$B$43:$C$57	Kirnberger col. 1	{"Master Frequenc...	
Master_Frequencies	='Kirnberger col. 2'!$B$43:$C$57	Kirnberger col. 2	{"Master Frequenc...	
Master_Frequencies	='Kirnberger col. 3'!$B$43:$C$57	Kirnberger col. 3	{"Master Frequenc...	
Master_Frequencies	='Kirnberger III P.'!$B$43:$C$57	Kirnberger III P.	{"Master Frequenc...	
Master_Frequencies	='Kirnberger III S.'!$B$43:$C$57	Kirnberger III S.	{"Master Frequenc...	
Master_Frequencies	='Meantone Bav Donat'!$B$43:$C$57	Meantone Bav Donat	{"Master Frequenc...	

Figure_Apx B-2 Name Manager



Figure_Apx B-1 Named Cell Ranges in Instrument Worksheet

This can be checked with the Name Manager in the Formulas menu as shown in Figure_Apx B-2.

B.3.2 Graphs

The graphs in any one sheet should all have the same y-axis scale for easy comparison. The scales are set by selecting the y-axis – right-click and “Format Axis …” and in the side bar, in Axis Options, set the bounds Minimum and Maximum, and further down under “Number” set the decimal places to zero.

B.4 Temperament Sheets

B.4.1 Units Selection

The units for tempering the fifths can be selected as Pythagorean Comma, Syntonic Comma or Cent, or fractions thereof 1/N, e.g. 1/4, 1/12.

The values for the drop-down list are defined as a named area “Units” (with workbook scope) in the ‘Commas and Cents’ sheet cells Y1-Y3. The drop-down list itself is then defined with the Excel Data Validation menu.

Note that some calculations are different depending on whether commas or cents are used.

B.4.2 Master Frequencies

To reduce problems when changing the layout of sheets, all the temperament sheets have a master table of the frequencies of the 12 notes of the scale at the bottom left of the sheet. This is named as "Master_Frequencies" and is used wherever the frequencies of the notes occur in formulae.

B.4.3 Adding an Invented Temperament Sheet

Copy a Template sheet and rename the sheet.

Add it to the indexes in the Overview sheet.

If copying from another temperament sheet, remember:

1) Check "Create a copy" in the Move/Copy window.
2) Rename the sheet.
3) Set the tab colour according to section B.1.
4) Change the title at the top left.
5) Set the required units (Syntonic or Pythagorean Comma or Cents) in cell D2 and the fraction of a unit in D3.
6) Set the frequency of A in cell I2
7) Enter the relative comma or cent values in cells C10 – C22, or for the 'Template from ET' in cells D10 – D22 using the selected units.
8) Adjust the y-axis scale of the graph "Difference to ET…" if necessary.
9) Remove or edit any Notes on cells (small red triangle in the top right corner of the cell).
10) Optionally add a text box to give the source and any other explanations.

B.4.4 Starting Note of the Scale

The sheets are all laid out to start from the note C. To start from a different note, e.g. A, all the sheets must be changed and the order of the notes in all the tables adapted accordingly. The last note must be the octave of the first, e.g. "a oct" instead of "c oct.".

The "Octaves down" column cells must also be adapted to bring the notes' frequencies into the same octave.

All the sheets must use the same note ordering so that the comparisons will work.

B.4.5 Calculation of C for a Given A

The frequency of the note C is calculated from the given frequency of A as follows:

For Organ: $C = A * L2/L1$ where L1 and L2 are the lengths of C and A.

For Scale: $C = A/2^{\frac{9}{12}}$, i.e. 9 semitones down from A.

For Fifths: $C = A * 2 * \frac{8}{27} * comma^{\Delta}$ where A is the frequency of A, comma is the fraction of a comma and Δ is the cumulative differences between A and C.

For ET: $C = \left(A/2^{9/12}\right) * 2^{\Delta/1200}$ where Ais the frequency of A and Δ is the cumulative differences between C and A in the given units.

B.5 The 'Best Fit Fretting' Worksheet

The first section contains the data imported from the BDO list of temperaments. The values are extracted from the links by using the developer view of the web page, copying the html and using a text editor to delete everything except the names and values. These are then pasted into the worksheet and the notes added in the row above them.

The instrument sheet to be tested is entered (best by copy and paste from the worksheet tab name) in the orange cell.

The next section has the fret notes in two heading rows obtained from the instrument sheet by transposing the columns to rows. These must be named in each instrument sheet with worksheet scope: "Frets_Note1", "Frets_Note2" and "Frets_Size".

The intervals are then calculated from the temperament from the equal temperament interval of 100 cents (this assumes that the fret is a semitone):

$$tempered\ interval = 100 + \Delta 1 - \Delta 2$$

Where Δ1 is the tempering of the first note and Δ2 is that of the second note, both in cents.

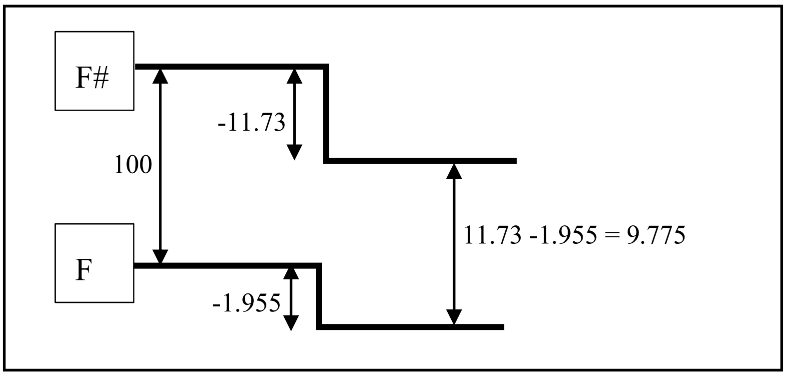


Figure 7-1 Resulting Interval of Tempered Fret

The next section gathers the intervals of the frets from the instrument as described above. The table calculates the difference between the instrument interval and the tempered interval.

The final column calculates the RMS of each row, and the minimum RMS is shown at the top.

It is important that the fret areas of the instrument sheets are named correctly as this ensures that no extra data is included in the heading rows. This means that unused cells have "#N/A" and they are filtered out of the RMS calculation; if they were zeroes, they would falsify the value.

To extend the tables great care must be taken that the formulae are filled correctly, taking into account the $-signs in the cell references that ensure that the appropriate row or column is fixed depending on whether it is filling across or down. (Excel mostly gets it right, but it must be verified.)

If and error "#Spill" is shown instead of the horizontal list of intervals, it means that there are not enough empty cells to the right to accommodate all the values from the instrument worksheet. In this case, columns must be inserted and the tables extended (see above).

B.6 The 'Best Fit Temperament' Worksheet

This is similar to the 'Best Fit Fretting' worksheet (section B.5 above), but takes the differences to Equal Temperament from the given temperament worksheet using the named ranges

"ET_Scale_Fifths" and "ET_Scale_Diff" to directly create a table of the differences between the BDO temperament and the given temperament and show the RMS values.

B.6.1 The 'BDO Any' Worksheet

This is based on the 'Template from Fifths' and looks up the differences in cents from the [BDO] data in the 'Best Fit Fretting' worksheet.

Since these are given in cents, the units cannot be changed.

The selected temperament name is reproduced in the instrument worksheets so that it is clear which temperament is being used (changing the selection in the 'BDO Any' sheet will be reflected in any instrument sheet that uses it).

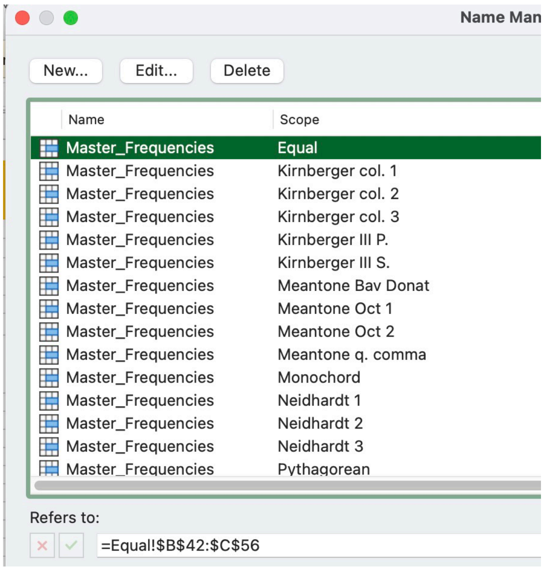
Appendix C Validation

C.1 By the User

As it is very easy for errors to creep in to complex Excel sheets, the user must check the results for plausibility.

The following checks should specifically be performed.

- Check that all formulae in a column use the correct next cell and the correct fixed cells (e.g. $A$1). (Excel will often show a small triangle in the top left corner of cells that do not follow the sequence.) In some places there is an intentional difference, e.g. to double a frequency to get the octave.
- Check that all sheets are linked in the Overview and that every sheet has a link back to the Overview.
- Check that all cells have the correct colour coded background.
- Check that all graphs are linked to the correct data – if a chart is copied from one sheet to another it may still show the data from the first sheet!
- Check that the Master_Frequencies, ET_Scale_Fifths and ET_Scale_Diff named areas are present for each temperament sheet with scope of that sheet.
- Check that the Frets_Note1, Frets_Note2and Frets_Size are named in each instrument worksheet with scope of that sheet.
- Check that the only named sections with workbook scope are BDO_Names, BDO_Table in the 'Best Fit Fretting' worksheet and Units in the 'Commas and Cents' worksheet.
- Check that all sheets are protected (Select the Review ribbon and use Cmd/Ctrl-right-arrow to move through the sheets and check that each has a padlock symbol.



C.2 Test Data

There is an Excel file Test_Data.xlsx which contains the temperament data for Werckmeister III in various forms. These can can be copied into the templates and the units set accordingly to show that they all give the same results.

C.3 Source Comparison

There is an Excel file Source_Comparison.xlsx which contains worksheets with the data for Werckmeister III from various sources to show that they are equivalent (or not!) – see section 1.5.